Notes on Cardinality

Definitions: Let A and B be non-empty sets.

A has the same cardinality as B if, and only if,
there exists a one-to-one correspondence from A to B, that is,

there exists a function $f: A \to B$ such that f is a one-to-one correspondence, that is,

there exists a function $f: A \to B$ such that f is one-to-one and onto.

As discussed below,
B has the same cardinality as A if, and only if, A has the same cardinality as B.

Thus, it is unambiguous to use the phrase "A and B have the same cardinality" to mean both, that B has the same cardinality as A and A has the same cardinality as B.

Observations: (Theorem 7.4.1)

A) For all sets A, A has the same cardinality as A. (Reflexive Property)

Proof: Using i_A, the identity function of set A, $i_A : A \to A$ is a one-to-one correspondence from A to A. Thus, A has the same cardinality as A.

B) For all sets A and B,
if A has the same cardinality as B, then B has the same cardinality as A.
(Symmetric Property)

Proof: Since A has the same cardinality as B, there exists a function $f: A \to B$ such that f is a one-to-one correspondence. Its inverse function $f^{-1} : B \to A$ is a one-to-one correspondence from B to A. Thus, B has the same cardinality as A.

C) For all sets A, B, and C,
if A has the same cardinality as B and B has the same cardinality as C,
then A has the same cardinality as C. (Transitive Property)

Proof: Since A has the same cardinality as B and B has the same cardinality as C, there exist functions $f: A \to B$ and $g: B \to C$ such that f and g are both one-to-one correspondences. Therefore, the composition $g \circ f: A \to C$ is a one-to-one correspondence. Thus, A has the same cardinality as C.

2)

Definitions:

A non-empty set B is <u>finite</u> if, and only if, there exists a positive integer n and a function f: { 1 , 2 , 3 , . . . , n } → B such that f is a one-to-one correspondence.

In this case, we say that n is the "number of elements in set B;" we say "the cardinality of the set B is n ;" and we say "B has the same cardinality as { 1 , 2 , 3 , . . . , n } ."

A non-empty set is <u>Infinite</u> if, and only if, it is not finite.

A non-empty set B is <u>Countably Infinite</u>

if, and only if, B has the same cardinality as $\mathbb{Z}^+$, that is,

there exists a function f: $\mathbb{Z}^+ \to$ B such that f is a one-to-one correspondence.

A non-empty set B is <u>Countable</u>

if, and only if, B is finite or countably infinite.

A set B is <u>Uncountable</u> if and only if B is not countable, that is,
if and only if B is infinite and not countably infinite.

The empty set ∅ will be considered to be a finite set.

Theorem (NIB) 11:

Given any non-empty countable set A, it is possible to <u>list</u> all of the elements of A in a sequence $a_1, a_2, a_3, \ldots, a_n$ (in the case that A is finite) or in a or in a sequence $a_1, a_2, a_3, \ldots$ (in the case that A is infinite) such that every element of A appears in the sequence once and only once.

Thus, for any countable set A, such a listing of the elements of A is possible.

Proof: Suppose A is any non-empty countable set

Thus, A is finite or A is infinite.

Case 1 (A is finite):

Suppose A is a finite set.

Then, by definition the of "finite set", there exists a positive integer n and a function f: { 1 , 2 , 3 , . . . , n } → A such that f is a one-to-one correspondence.

Define a_1 = f(1), a_2 = f(2), a_3 = f(3), . . . , a_n = f(n) .

Then, the sequence $a_1, a_2, a_3, \ldots, a_n$ has no repetitions (since f is one-to-one) and the sequence includes all the elements of A (since f is onto).

Thus, such a listing of the elements of A is possible in the case that A is a finite set.

Case 2 (A is infinite) :

Suppose A is an infinite set.

∴ As an infinite countable set, A is countably infinite.

∴ There exists a function f: $\mathbb{Z}^+ \rightarrow A$ such that f is a one-to-one correspondence.

Define $a_1 = f(1)$, $a_2 = f(2)$, $a_3 = f(3)$, ...

Then, the sequence $a_1, a_2, a_3, \ldots$ has no repetitions (since f is one-to-one) and the sequence includes all the elements of A (since f is onto).

Thus, such a listing of the elements of A is possible in the case that A is an infinite set.

Therefore, such a listing of the elements of A is possible in general.

Q E D, by Direct Proof.

Note: In a writing a proof involving a set B which is known to be finite or countably infinite, it is useful to say early on in the proof:

(If B is finite with n elements)

"Since B is finite with n elements, the elements of B can be listed in a finite sequence $b_1, b_2, b_3, \ldots, b_n$ by Theorem (NIB) 11.

OR

(If B is countably infinite)

"Since B is countably infinite, the elements of B can be listed in an infinite sequence $b_1, b_2, b_3, \ldots$ by Theorem (NIB) 11."

This means that every element of B appears in the sequence and no element of B appears more than once.

This will allow us to discuss the elements of B individually, which is useful in certain instances, such as, for example, in defining functions from $\mathbb{Z}^+$ to B and defining functions from B to $\mathbb{Z}^+$.

For example:

"Suppose that B countably infinite. Since B is countably infinite, the elements of B can be listed in an infinite sequence $b_1, b_2, b_3, \ldots$.

Define function f: $B \rightarrow \mathbb{Z}^+$ as follows: For all $n \in \mathbb{Z}^+$, define $f(b_n) = 6 + 2n$.

It can be shown that f is one-to-one but not onto."

Example 1: We show that the sets $\mathbb{Z}$ and $2\mathbb{Z}$ have the same cardinality, where

$2\mathbb{Z}$ is the set $2\mathbb{Z} = \{ n \in \mathbb{Z} \mid n = 2k \text{ for some integer } k \} = \{ \text{all EVEN integers} \}$.

Proof: Define the function $f: \mathbb{Z} \to 2\mathbb{Z}$ as follows: For all $k \in \mathbb{Z}$, $f(k) = 2k$.

Then, define the function $g: 2\mathbb{Z} \to \mathbb{Z}$ as follows:

For all $n \in 2\mathbb{Z}$, $g(n) = \frac{1}{2}n$.

[Note: $\frac{1}{2}n \in \mathbb{Z}$: Since $n \in 2\mathbb{Z}$, $n = 2k$ for some integer k, so $\frac{1}{2}n = k \in \mathbb{Z}$.]

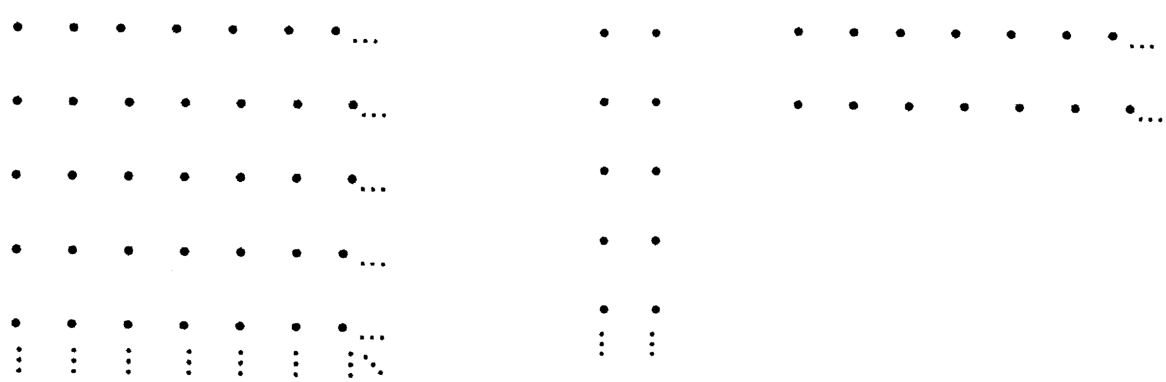
For all $k \in \mathbb{Z}$, $g \circ f(k) = g(f(k)) = \frac{1}{2}f(k) = \frac{1}{2}(2k) = k$.

For all $n \in 2\mathbb{Z}$, $f \circ g(n) = f(g(n)) = 2(g(n)) = 2(\frac{1}{2}n) = n$.

Therefore, by Theorem (NIB) 10, f is a one-to-one correspondence.

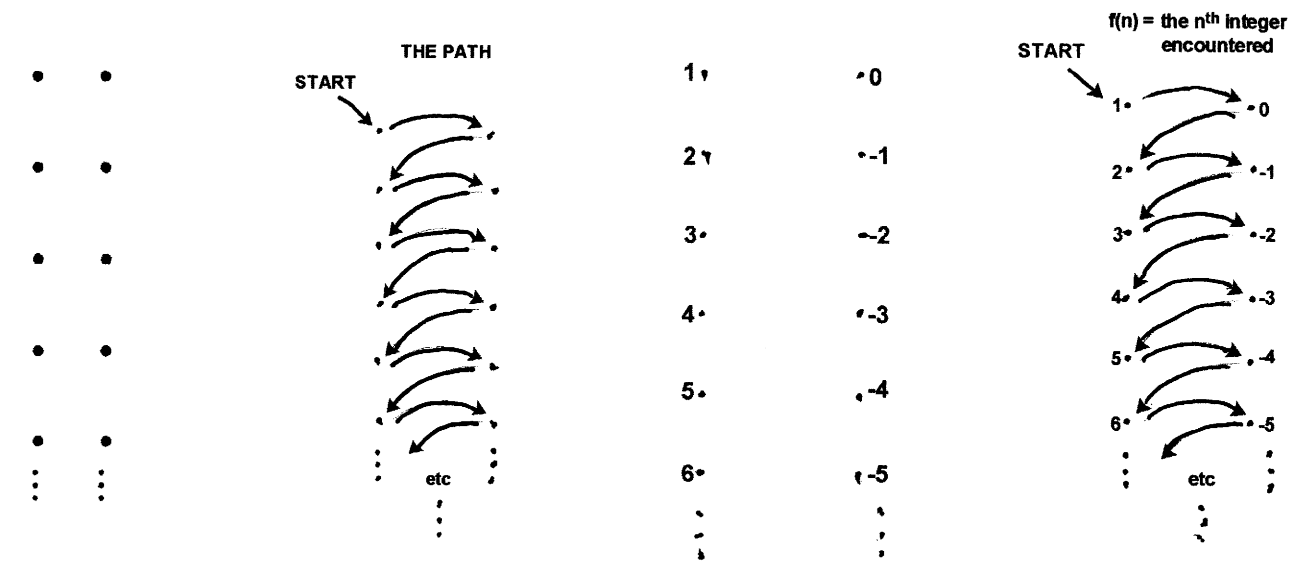
$\therefore$ $\mathbb{Z}$ and $2\mathbb{Z}$ have the same cardinality. QED

Definition: A <u>lattice</u> is a finite or infinite collection of discrete points of the plane arranged in rows and columns. A standard example is the collection of points (x , y) in the Cartesian Coordinate Plane such that both coordinates x and y are integers. Even a single row or a single column from this collection is a lattice also.



Defining a Function f: $\mathbb{Z}^+ \rightarrow$ **A SET** in terms of a Path through a Lattice

We use a method of defining a function that is unconventional, but still it is a valid method for defining a function. We define a function in terms of a **path** through a lattice. Consider the first lattice below. We define a path through the lattice by indicating where the path begins and a systematic way the path proceeds so that every point in the lattice is eventually visited along the path exactly once. (See the second lattice below. Note that, when each point is visited, only a finite number of points have been visited previously!



We will define a function f from $\mathbb{Z}^+$ to $\mathbb{Z}$ using this unconventional method. First, we place all of the elements of the **co-domain** set on the points of the lattice which we have chosen to use. You can think of the placement process as a **labeling** of the points with elements of the co-domain of the function f. Here, f: $\mathbb{Z}^+ \rightarrow \mathbb{Z}$, so we systematically place each integer from $\mathbb{Z}$ on some point in the lattice. One such placement scheme is shown on the lattice above.

The Defining of the Function:

Place the integers in $\mathbb{Z}$ on the lattice points as shown above and then traverse the lattice along THE PATH indicated.

Define function $f : \mathbb{Z}^+ \rightarrow \mathbb{Z}$ as follows:

For all $n \in \mathbb{Z}^+$, f(n) = the n^{th} integer in $\mathbb{Z}$ that is encountered along THE PATH through the lattice as shown above.

Thus, f(1) = 1 , f(2) = 0 , f(3) = 2 , f(4) = -1 , f(5) = 3 , f(6) = -2 , etc.

In this manner, function f is well-defined. For each positive integer n, the function value f(n) has been defined and can actually be known if we just travel down the path far enough.

Example 2:

To Prove: The set of integers, $\mathbb{Z}$, is countably infinite.

Proof: A set B is countably infinite if and only if there exists a function $f : \mathbb{Z}^+ \rightarrow B$ such that f is a one-to-one correspondence, that is, f is one-to-one and onto.

We use the function f defined on the previous page. Rather than re-write the definition of f here again, we refer the reader to the previous page to find the definition there.

Note that, regarding the placement scheme used in the definition of f, every integer from $\mathbb{Z}$ has been placed on a point of the lattice and has been placed on only one point of the lattice. Thus, in traversing the lattice along THE PATH, each integer is encountered at least once and no integer is ever encountered twice.

Thus, function f is onto, because every integer from $\mathbb{Z}$ is encountered at least once along THE PATH, and the function f is one-to-one because no integer is ever encountered more than once.

Therefore, f is a one-to-one correspondence from $\mathbb{Z}^+$ to $\mathbb{Z}$.

Therefore, $\mathbb{Z}^+$ and $\mathbb{Z}$ have the same cardinality and so $\mathbb{Z}$ is countably infinite . QED

The definition of the function f in this proof might seem suspect, but it is not. It might also be unsatisfying because the definition does not use a formula to calculate the value f(n) for any given value of n. Often, for functions f defined in this unconventional way, there do exist formulas that calculate the value of f(n). However, for the function f defined in the proof above, there does exist a formula that calculates f(n) for any given n.

Define function $f : \mathbb{Z}^+ \rightarrow \mathbb{Z}$ as follows: For all $n \in \mathbb{Z}$,

$$f(n) = \begin{cases} \frac{1}{2}(n+1) & \text{if } n \text{ is odd} \quad (\text{Note: Here, } f(n) > 0.) \\ -(\frac{1}{2}n) + 1 & \text{if } n \text{ is even.} \quad (\text{Note: Here, } f(n) \leq 0.) \end{cases}$$

Thus, f(1) = 1 , f(2) = 0 , f(3) = 2 , f(4) = -1 , f(5) = 3 , f(6) = -2 , etc.

Function f, so defined by formula, is the same function f defined before using the unconventional method. It is simple to write proofs verifying that f is a one-to-one correspondence. Therefore, again we see, $\mathbb{Z}^+$ and $\mathbb{Z}$ have the same cardinality and so $\mathbb{Z}$ is countably infinite. QED Note: This also means that $\mathbb{Z}$ is a countable set.

Theorem 7.4.3: Any subset of a countable set is countable.

Proof: Note that every subset of a finite set is also a finite set. It is impossible for a finite set to contain an infinite subset.

Suppose A is any countable set. Let B be any subset of A, that is, $B \subseteq A$.

Any subset of a finite set is a finite set and is thus countable, by definition of "countable". Therefore, without loss of generality, **we assume A is infinite.**

As an infinite countable set, A is countably infinite. [NTS: B is countable]

The set B is finite or the set B is infinite.

Case 1 (B is finite):

Suppose B is finite. Therefore, B is countable, by definition of "countable set".

∴ B is countable, in Case 1.

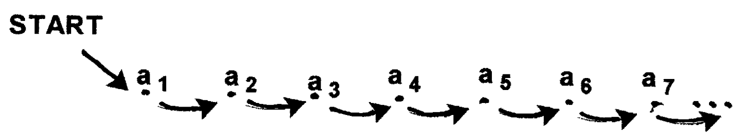
Case 2 (B is infinte):

Suppose that B is an infinite set.

Since A is countably infinite, the elements of A can be listed in an infinite sequence $a_1, a_2, a_3, \ldots$ by Theorem (NIB) 11, such that every element of A appears in the sequence once and only once.

Therefore, every element of B appears in the sequence once and only once.

On a lattice consisting of one line, place the elements of A on the lattice points as shown below and then traverse the lattice along the path indicated here.



Define function $f: \mathbb{Z}^+ \to B$ as follows:

For all $n \in \mathbb{Z}^+$, $f(n)$ = the n^{th} element in set B that is encountered along the path through the lattice as shown above.

Function f is well-defined because every element of B appears in the sequence at least once and, since there are infinitely many elements of B to encounter, for every positive integer n, there is an n^{th} element of B that is encountered.

The function f is onto because every element of B is encountered at least once.

The function f is one-to-one because no element of B is encountered more than once.

Therefore, f is a one-to-one correspondence from $\mathbb{Z}^+$ to B.

∴ B has the same cardinality as $\mathbb{Z}^+$. ∴ B is countably infinite.

∴ B is countable by the definition of "countable", in Case 2.

∴ B is countable in general.

∴ Any subset of a countable set is countable, by Direct Proof. QED

In Theorem (NIB) 12 below, it is proved the union of two countable sets is a countable set. We will need to prove two Lemmas in preparation for the proof of Theorem (NIB) 12.

Lemma #1 for Theorem (NIB) 12:

The union of two disjoint countable sets, at least one of which is a finite set, is a countable set.

Proof:

Let A and B be two disjoint countable sets, at least one of which is a finite set.

[NTS: $A \cup B$ is countable]

Recall: A and B are disjoint sets if and only if $A \cap B = \varnothing$.

Without loss of generality, we assume that set A is a finite set, with k elements, say.

Since A is finite with k elements, the elements of A can be listed in a finite sequence $a_1, a_2, a_3, \ldots, a_k$ by Theorem (NIB) 11.

B is finite or B is infinite.

Case 1 (B is finite):

Suppose B is finite. Then, $A \cup B$ is also finite.

Therefore, $A \cup B$ is countable, by definition of "countable", in Case 1.

Case 2 (B is infinite):

Suppose B is infinite.

As an infinite countable set, B is countably infinite.

Since B is countably infinite, the elements of B can be listed in an infinite sequence $b_1, b_2, b_3, \ldots$ by Theorem (NIB) 11.

On a lattice consisting of one line, place the elements of $A \cup B$ on the lattice points as shown below, placing the elements of A in front of the elements of B.

Then, traverse the lattice along the path indicated here.

Define function $f : \mathbb{Z}^+ \to A \cup B$ as follows:

For all $n \in \mathbb{Z}^+$, $f(n)$ = the n^{th} element in the set $A \cup B$ that is encountered along the path through the lattice as shown above.

Function f is onto because every element of $A \cup B$ is encountered along the path at least once.

Since $A \cap B = \varnothing$, no element of A is encountered among the elements of B and no element of B is encountered among the elements of A. Therefore, function f is one-to-one since no element of $A \cup B$ is encountered along the path more than once.

Therefore, f is a one-to-one correspondence from $\mathbb{Z}^+$ to $A \cup B$.

$\therefore$ $A \cup B$ has the same cardinality as $\mathbb{Z}^+$. $\therefore$ $A \cup B$ is countably infinite.

$\therefore$ $A \cup B$ is countable by the definition of "countable", in Case 2.

$\therefore$ $A \cup B$ is countable in general.

$\therefore$ The union of two disjoint countable sets, at least one of which is a finite set, is a countable set, by Direct Proof. QED

Lemma #2 for Theorem (NIB) 12:

The union of two disjoint countable sets, both of which are countably infinite, is a countable set.

Proof:

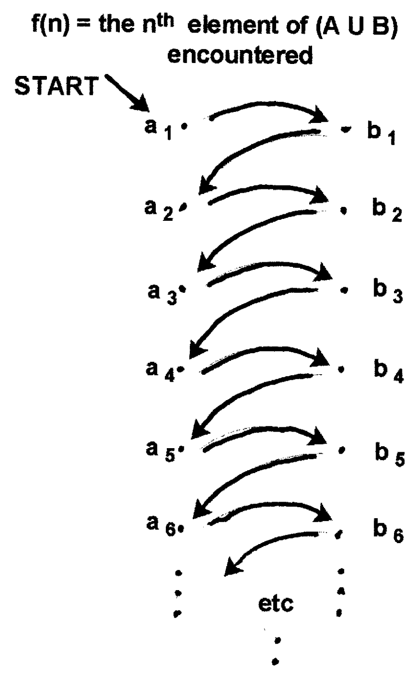
Let A and B be two disjoint countably infinite sets. [NTS: $A \cup B$ is countable]

Recall: A and B are disjoint sets if and only if $A \cap B = \varnothing$.

Since A and B are both countably infinite, the elements of A can be listed in an infinite sequence $a_1, a_2, a_3, \ldots$ and the elements of B can be listed in an infinite sequence $b_1, b_2, b_3, \ldots$ by Theorem (NIB) 11.

On a lattice consisting of two lines, place the elements of $A \cup B$ on the lattice points as shown here, placing the elements of A on the points of the left-hand column and placing the elements of B on the points of the right-hand column.

Then, traverse the lattice along the path indicated here.



Define function $f: \mathbb{Z}^+ \to A \cup B$ as follows:

For all $n \in \mathbb{Z}^+$, f(n) = the n^{th} element in the set $A \cup B$ that is encountered along the path through the lattice as shown here.

Function f is onto because every element of $A \cup B$ is encountered along the path at least once.

Since $A \cap B = \varnothing$, no element of A is encountered among the elements of B and no element of B is encountered among the elements of A. Therefore, function f is one-to-one since no element of $A \cup B$ is encountered along the path more than once.

Therefore, f is a one-to-one correspondence from $\mathbb{Z}^+$ to $A \cup B$.

$\therefore$ $A \cup B$ has the same cardinality as $\mathbb{Z}^+$. $\therefore$ $A \cup B$ is countably infinite.

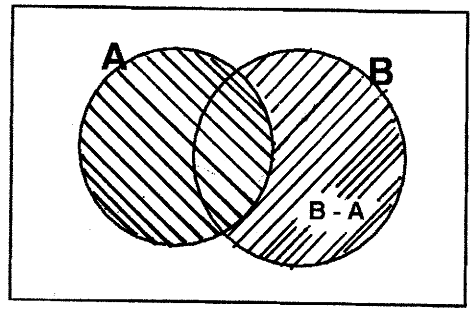
$\therefore$ $A \cup B$ is countable by the definition of "countable".

$\therefore$ The union of two disjoint countably infinite sets is a countable set, by Direct Proof. QED

Theorem (NIB) 12: The union of any two countable sets is a countable set.

Proof: Suppose A and B are countable sets.

[NTS: $A \cup B$ is countable]



$A \cup B = A \cup (B - A)$.

Also, A and (B - A) are disjoint sets, that is,

$A \cap (B - A) = \varnothing$.

Since $(B - A) \subseteq B$ and B is countable,

(B - A) is countable, by Theorem 7.4.3.

One of the following is true:

(1) Both sets A and (B - A) are finite;

(2) One of the two sets is finite and the other is infinite;

(3) Both sets A and (B - A) are infinite.

Case 1 (Both sets A and (B - A) are finite):

Suppose both sets A and (B - A) are finite.

Then, $A \cup (B - A)$ is finite.

∴ $A \cup (B - A)$ is countable by the definition of "countable", in Case 1.

Case 2 (One of the two sets is finite and the other is infinite):

Suppose one of the two sets is finite and the other is infinite.

∴By Lemma #1 for this theorem, $A \cup (B - A)$ is countable, in Case 2.

Case 3 (Both sets A and (B - A) are infinite):

Suppose both sets A and (B - A) are infinite.

Then, being both infinite and countable, both sets A and (B - A) are countably infinite.

∴By Lemma #2 for this theorem, $A \cup (B - A)$ is countable in Case 3.

Therefore, $A \cup (B - A)$ is countable in general.

Since $A \cup B = A \cup (B - A)$, $A \cup B$ is countable, by substitution.

∴ The union of any two countable sets is a countable set, by Direct Proof. QED

Corollary 7.4.4:

For any set A, if A contains an uncountable subset B, then A is uncountable.

Proof: Suppose A is a set such that A contains a subset B such that B is uncountable.

[NTS: A is uncountable.]

Suppose, by way of contradiction, that A is countable.

∴ Since $B \subseteq A$, B is countable, by Theorem 7.4.3, which contradicts the fact that B is uncountable.

∴ A is uncountable, by proof-by-contradiction.

∴ For any set A, if A contains an uncountable subset B, then A is uncountable, by Direct Proof.

QED

Using Theorem 7.4.3, we can prove that particular infinite subsets of the set of Integers are countable sets because the set of Integers is a countable set.

Thus, we conclude that the set of all odd integers is countable, by Theorem 7.4.3 and the fact that it is a subset of the countable set $\mathbb{Z}$.

Similarly, the set of all prime numbers is countable.

Theorem (NIB) 13: $\mathbb{Q}^+$, the set of positive rational numbers, is countably infinite.

Proof: Clearly, $\mathbb{Q}^+$ is infinite since $\mathbb{Q}^+$ contains the set of positive integers.

To define a one-to-one correspondence from $\mathbb{Z}^+$ to $\mathbb{Q}^+$,

place the *representations* of the positive rational numbers on a lattice quadrant, as shown in Example 7.4.4 on page 338, Figure 7.4.3, as also shown below:

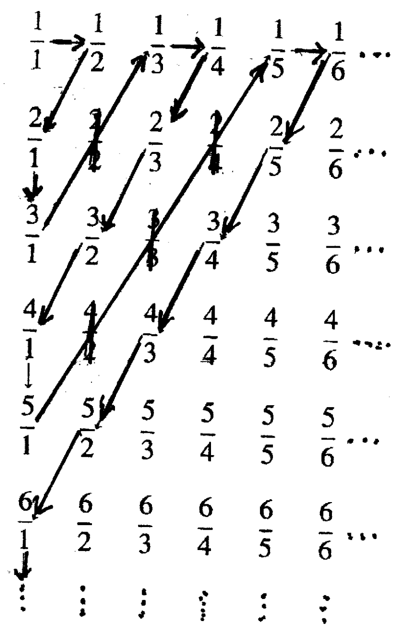


Figure 7.4.3

Traverse the lattice along the path indicated above.

Define $f: \mathbb{Z}^+ \to \mathbb{Q}^+$ as follows: For all $n \in \mathbb{Z}^+$,

f(n) = the n^{th} <u>**newly encountered**</u> positive rational number,
encountered along the path through the lattice indicated above.

Function f is one-to-one because previously-encountered rational numbers are skipped over.

Function f is onto because
every positive rational number eventually is newly-encountered along the path.

Therefore, f is a one-to-one correspondence from $\mathbb{Z}^+$ to $\mathbb{Q}^+$.

$\therefore$ $\mathbb{Q}^+$ is countably infinite. Q E D

Note: This also proves that $\mathbb{Q}^+$ is a countable set, by definition of "countable".